Nonassociativity, Dirac monopoles and Aharonov-Bohm effect

A. I. Nesterov*

Departamento de Física, CUCEI, Universidad de Guadalajara,
Av. Revolución 1500, Guadalajara, CP 44420, Jalisco, México
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The Aharonov-Bohm (AB) effect for the singular string associated with the Dirac monopole carrying an arbitrary magnetic charge is studied. It is shown that the emerging difficulties in explanation of the AB effect may be removed by introducing nonassociative path-dependent wave functions. This provides the absence of AB effect for the Dirac string of nonquantized magnetic monopole.

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Nowadays interest in the Dirac monopole problem [1] is growing in connection with the 'fictitious' monopoles that are similar to the 'real' magnetic monopoles, however, appearing in the context of the Berry phase [2]. Recently, the experimental results providing evidence for the 'fictitious' magnetic monopole in the crystal-momentum space has been reported in relation to the anomalous Hall effect [3]. Besides of the anomalous Hall effect, these type of magnetic monopoles emerges in trapped Λ-type atoms, anisotropic spin systems, noncommutative quantum mechanics, etc., and *may carry an arbitrary 'magnetic' charge* [3, 4, 5, 6, 7, 8, 9, 10].

It is known that the Dirac quantization condition

$$2\mu = n, \; n \in \mathbb{Z},$$

where $\mu = eq$, electric charge being e and magnetic charge q (we set $\hbar = c = 1$), does not follow from the quantum-mechanical consideration alone. Any treatment uses some additional assumptions that may be not physically inevitable. For instance, in Dirac's theory 'quantization of magnetic charge' follows from the requirement of the wave function be single-valued. However, single-valudness is not one of the fundamental principles of quantum mechanics, and having multi-valued wave functions may be allowed until it does not affect the algebra of observables.

Other well-known topological and geometrical arguments in behalf of Dirac quantization rule is based on employing classical fibre bundle theory [11, 12, 13, 14, 15, 16, 17]. In this approach the Dirac monopole is treated as the Hopf bundle $U(1)$ over S^2. A realization of the Dirac monopole in this way implies that there exists the division of space into overlapping regions $\{U_i\}$ with nonsingular vector potential being defined in $\{U_i\}$ and yielding the correct monopole magnetic field in each region. On the triple overlap $U_i \cap U_j \cap U_k$ it holds

$$\exp(i(q_{ij} + q_{jk} + q_{ki})) = \exp(i4\pi\mu) \qquad (1)$$

where q_{ij} are the transition functions such that $U_i \cap U_j \rightarrow U(1)$. The consistency condition, which is equivalent to the associativity of the group multiplication, requires $q_{ij} + q_{jk} + q_{ki} = 0 \mod 2\pi\mathbb{Z}$. This gives rise to the Dirac selectional rule $2\mu \in \mathbb{Z}$ as a necessary condition to have a consistent $U(1)$-bundle over S^2 [11, 12, 13].

Finally, group-theoretical approach requires Dirac's quantization as a necessary condition that the operator of total angular momentum generates a finite-dimensional representation of the rotation group [18, 19, 20, 21, 22, 23].

At the first sight these results restrict the magnetic monopole's world by the monopoles which just satisfy the Dirac quantization condition. Nevertheless, the self-consistent theory of Dirac's monopole with an arbitrary magnetic charge can be constructed nonassociative structures such as quasigroups and loops [24, 25, 26, 27]. To this end one needs to consider the *nonassociative* generalization of $U(1)$ bundle over S^2 employing nonassociative fibre bundle theory [24, 25, 26, 27], and in the context of the group theory one has to consider *infinite-dimensional representations* of the rotation group [27, 28, 29]

As is well known, any choice of the vector potential yielding a magnetic field of the Dirac monopole must have singularity, known as the Dirac string. In this the exploration of the Aharonov-Bohm [30] (AB) effect for the Dirac singular string has been of great interest. The AB effect is appeared in quantum interference between two parts of a beam of charged particles, say electrons with charge e, passing by an infinite long solenoid. The interference pattern on the screen does not change if the relative phase shift $\Delta\varphi = e\Phi$, where Φ is the total magnetic flux through solenoid, satisfies the following condition: $\Delta\varphi = 2\pi n$, $n \in \mathbb{Z}$.

What makes difference between the infinite long solenoid and Dirac's string is that the latter can be moved out of position by a singular gauge transformation. This means that the monopole string can not be observed in AB experiment (if singular gauge transformations are allowed). Thus, the absence of AB effect for the Dirac string has a crucial significance for a consistent magnetic monopole theory, and, as it is well-known, this require-

ment yields the Dirac quantization condition.

This raises the question of whether the absence of AB effect for the Dirac string is compatible with arbitrariness of magnetic monopole charge. In this Letter we show that the response is affirmative, but it requires employment of nonassociative structures like quasigroups and loops.

Magnetic monopole preliminaries. – Since any choice of the vector potential $\mathbf{A}$ being compatible with a magnetic field $\mathbf{B} = q\mathbf{r}/r^3$ of Dirac monopole must have the singular string, one has $\mathbf{B} = \nabla \times \mathbf{A}$ locally, but not globally, and this implies

$$\mathbf{B} = \mathrm{rot}\mathbf{A} + \mathbf{h} \tag{2}$$

where $\mathbf{h}$ is the magnetic field of the Dirac string $\mathcal{C}$.

There is an ambiguity in the definition of the vector potential. For instance, Dirac introduced the vector potential as [1]

$$\mathbf{A_n} = q\frac{\mathbf{r}\times\mathbf{n}}{r(r-\mathbf{n}\cdot\mathbf{r})} \tag{3}$$

where the unit vector $\mathbf{n}$ determines the direction of string (hereafter denoted by $S_\mathbf{n}$), which passes from the origin of coordinates to ∞ , and

$$\mathbf{h_n} = 4\pi q\mathbf{n}\int_0^\infty \delta^3(\mathbf{r}-\mathbf{n}\tau)d\tau. \tag{4}$$

Schwinger's choice is [31]:

$$\mathbf{A}^{SW} = \frac{1}{2}\big(\mathbf{A_n} + \mathbf{A_{-n}}\big), \tag{5}$$

with the string being propagated from $-\infty$ to ∞. Both vector potentials yield the same magnetic monopole field, however the quantization is different. The Dirac condition is $2\mu = p$, while the Schwinger one is $\mu = p,\ p \in \mathbb{Z}$.

These two strings belong a family $\{S^\kappa_\mathbf{n}\}$ of *weighted strings*, which magnetic field is given by [35]

$$\mathbf{h}^\kappa_\mathbf{n} = \frac{1+\kappa}{2}\mathbf{h_n} + \frac{1-\kappa}{2}\mathbf{h_{-n}} \tag{6}$$

where κ is the weight of a semi-infinite Dirac string. The respective vector potential reads

$$\mathbf{A}^\kappa_\mathbf{n} = \frac{1+\kappa}{2}\mathbf{A_n} + \frac{1-\kappa}{2}\mathbf{A_{-n}}. \tag{7}$$

Note that two arbitrary strings $S^\kappa_\mathbf{n}$ and $S^\kappa_{\mathbf{n}'}$ are related by gauge transformation:

$$A^{\kappa'}_{\mathbf{n}'} = A^\kappa_\mathbf{n} + d\chi., \tag{8}$$

and the corresponding wave function transforms as follows: $\psi' = \exp(-i\chi)\psi$. In fact, in Dirac's approach and subsequent development of the monopole theory the wave functions are considered as having a non-integrable (or path-dependent) phase factor [1, 11, 12, 32]

Aharonov-Bohm effect, Dirac's string and nonassociative algebra of observables. – Let a coherent beam of electrons with charge e is incoming from $-\infty$. The beam is split at the point P in two parts, passing by an infinite long solenoid (Fig.1). In spite of the fact that the magnetic field $\mathbf{B}$ outside the solenoid is equal to zero, it produces an interference effect at the point Q of the screen, where the beams are brought together [30].

Following Mandelstam [32], let us consider the path-dependent wave function

$$\Psi(\mathbf{r},\gamma) = e^{i\alpha_1(\mathbf{r},\gamma)}\psi(\mathbf{r}), \tag{9}$$

where $\psi(\mathbf{r})$ denotes the wave function in the absence of the magnetic field, $\mathbf{A} = 0$, and $\alpha_1(\mathbf{r},\gamma) = e\int_\mathbf{r}^{\mathbf{r}'}\mathbf{A}\cdot d\mathbf{r}$, the integration is performed along the path γ connecting $\mathbf{r}$ and $\mathbf{r}'$: $\mathbf{r} \xrightarrow{\gamma} \mathbf{r}'$.

The total wave function at the point Q of the screen is the superposition of the wave functions along the both paths

$$\begin{aligned}\Psi_Q &= \Psi_1(\mathbf{r},\gamma_1) + \Psi_2(\mathbf{r},\gamma_2)\\ &= e^{i\alpha_1(\mathbf{r},\gamma_1)}\psi_1(\mathbf{r}) + e^{i\alpha_1(\mathbf{r},\gamma_2)}\psi_2(\mathbf{r})\\ &= e^{i\alpha_1(\mathbf{r},\gamma_2)}\big(e^{ie\oint\mathbf{A}\cdot d\mathbf{r}}\psi_1(\mathbf{r}) + \psi_2(\mathbf{r})\big),\end{aligned} \tag{10}$$

where $\alpha_1(\mathbf{r},\gamma_1)$ and $\alpha_1(\mathbf{r},\gamma_2)$ are equal to $-e\int_P^Q\mathbf{A}\cdot d\mathbf{r}$ along the paths of the first and second beam respectively.

A relative phase shift $\Delta\varphi$ is given by

$$\Delta\varphi = e\oint_\gamma \mathbf{A}\cdot d\mathbf{r} = e\Phi, \tag{11}$$

where the integration ie performed along the closed path $\gamma = \gamma_1\cup\gamma_2$, Φ being the total magnetic flux through the solenoid. The condition for the absence of observable AB effect is $e\Phi = 2\pi n,\ n\in\mathbb{Z}$.

Let us assume that the beam passes in the upper half of the space divided by the plane $z = 0$. Then the contribution of the string $S^\kappa_\mathbf{n}$ to the relative phase shift of the wave function at the point Q is found to be

$$\Delta\varphi = 2\pi(1+\kappa)\mu \tag{12}$$

and, if the beam passes in the lower half-space ($z < 0$), one has

$$\Delta\varphi = 2\pi(\kappa-1)\mu \tag{13}$$

It follows from the Eqs. (12), (13) the absence of AB effect when $(1+\kappa)\mu$ and $(1-\kappa)\mu$, both are integers. In the case of $\kappa \neq 0$, this yields immediately the following conditions: $2\mu\in\mathbb{Z}$, that is the celebrated Dirac rule, and quantization of the string weight, $2\kappa\mu\in\mathbb{Z}$. If $\kappa = 0$, one obtains the Schwinger quantization condition, $\mu\in\mathbb{Z}$.

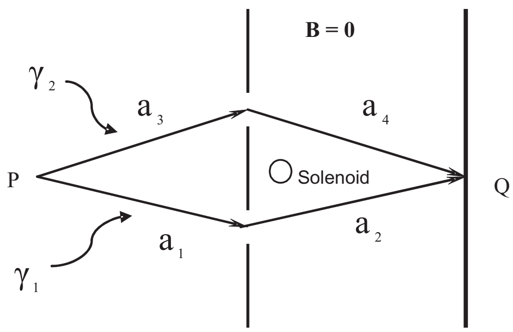

FIG. 1: The simplified scheme of the Aharonov-Bohm experiment. The magnetic field $\mathbf{B} = 0$ outside the solenoid.

At first sight the existence of the magnetic monopoles with an arbitrary magnetic charge is in contradiction with the AB experiment. To clarify issue let us recall that the relative phase shift (11) arises as result of the parallel translation of wave function along the contour γ surrounding the Dirac string. It is known that for the generators of translations the Jacobi identity fails and for the finite translations one has [13, 33]

$$\big(U_{\mathbf{a}}U_{\mathbf{b}}\big)U_{\mathbf{c}} = \exp(i\alpha_3(\mathbf{r};\mathbf{a},\mathbf{b},\mathbf{c}))U_{\mathbf{a}}\big(U_{\mathbf{b}}U_{\mathbf{c}}\big) \tag{14}$$

where α_s is the so-called *three cocycle*; $\alpha_3 = 4\pi\mu \mod 2\pi\mathbb{Z}$, if the monopole is enclosed by the simplex with vertices $(\mathbf{r}, \mathbf{r}+\mathbf{a}, \mathbf{r}+\mathbf{a}+\mathbf{b}, \mathbf{r}+\mathbf{a}+\mathbf{b}+\mathbf{c})$ and zero otherwise [33]. For the Dirac quantization condition being satisfied $\alpha_3 = 0 \mod 2\pi\mathbb{Z}$, and (14) provides an associative representation of the translations, in spite of the fact that the Jacobi identity continues to fail. Thus, we see that the AB effect requires more careful analysis, if one assumes existence of an arbitrary monopole charge.

The emerging difficulties in explanation of the AB effect may be removed by introducing nonassociative path-dependent wave function $\Psi(\mathbf{r};\gamma)$. Here we consider the simplified version of the approach developed in [24, 25, 26].

Let us start with the nonassociative generalization of the gauge transformations known as *gauge loop* $\mathcal{Q}$, which is related to the so-called *string group* [25, 34]. The string group, denoted as $\mathrm{String}\,\mathfrak{M}$, is the group of all paths γ: $[0,1] \mapsto \mathrm{Diff}\,\mathfrak{M}$, where $\mathrm{Diff}\,\mathfrak{M}$ denotes the diffeomorphism group on $\mathfrak{M} = R^3\backslash\{0\}$ such that $\mathbf{r} \mapsto \mathbf{r}(t) = \mathbf{r}\gamma(t)$, $t \in [0,1]$ and $\gamma(0) = \text{identity}$; the group composition is defined as $\gamma_{12}(t) = \gamma_1(t) \circ \gamma_2(t)$.

Now let f_γ be the map:

$$f_\gamma : \mathbf{r} \mapsto U_{g_\gamma} = \exp\big(i\alpha_1(\mathbf{r};\gamma)\big) \in \mathcal{Q}, \tag{15}$$

$$\alpha_1(\mathbf{r};\gamma) = e\int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{A}(\boldsymbol{\xi})\cdot d\boldsymbol{\xi}, \tag{16}$$

where the integration is performed along a path connecting a point $\mathbf{r}$ with a point $\mathbf{r}' = \mathbf{r}\gamma(1)$, and $\mathbf{A}(\mathbf{r})$ is the vector potential.

The product of two elements $U_{g_{\gamma_1}}, U_{g_{\gamma_2}} \in \mathcal{Q}$ is defined as follows:

$$\begin{aligned} &U_{g_{\gamma_1}} * U_{g_{\gamma_2}} = U_{g_{\gamma_1}\cdot g_{\gamma_2}},\ U_{g_{\gamma_1}\cdot g_{\gamma_2}} \in \mathcal{Q} \\ &g_{\gamma_1}\cdot g_{\gamma_2} = \alpha_1(\mathbf{r};\gamma_1) + \alpha_1(\mathbf{r}_1;\gamma_2) + \sigma(\mathcal{C},\Sigma)), \end{aligned} \tag{17}$$

where $\mathbf{r}_1 = \mathbf{r}\gamma_1(1)$, and $\sigma(\mathcal{C},\Sigma)$ being the contribution of the Dirac string:

$$\sigma = e\int_\Sigma \mathbf{h}\cdot d\mathbf{S} \tag{18}$$

is not zero if and only if the string $\mathcal{C}$ crosses the surface Σ. The surface is parameterized as $\mathbf{r}(t,s) = \mathbf{r}\gamma_1(t)\gamma_2(s)$ with $0 \le s \le t \le 1$, and the vertices are $(\mathbf{r},\mathbf{r}_1,\mathbf{r}_2)$, where $\mathbf{r}_1 = \mathbf{r}\gamma_1(1)$ and $\mathbf{r}_2 = \mathbf{r}\gamma_2(1)$.

For the triple-product we obtain the following result:

$$U_{g_{\gamma_1}} * (U_{g_{\gamma_2}} * U_{g_{\gamma_3}}) = \mathrm{e}^{i\alpha_3(\mathbf{r};\gamma_1,\gamma_2,\gamma_3)}(U_{g_{\gamma_1}} * U_{g_{\gamma_2}}) * U_{g_{\gamma_3}}$$

where $\alpha_3 = 4\pi\mu$ is the three-cocycle. As is easy to see, if the Dirac quantization condition is fulfilled, the operation $*$ is associative and the local loop $\mathcal{Q}$ becomes the gauge group $U(1)$.

The nonassociative path dependent wave function is given by

$$\Psi(\mathbf{r},\gamma) = \mathrm{e}^{i\alpha_1(\mathbf{r},\gamma)}\Psi(\mathbf{r}), \quad \mathrm{e}^{i\alpha_1(\mathbf{r},\gamma)} \in \mathrm{QU}(1), \tag{19}$$

and the realization of gauge loop $\mathcal{Q}$ in the space of the wave functions $\Psi(\mathbf{r},\gamma)$ is defined as follows:

$$U_{g_{\gamma_1}} * \Psi(\mathbf{r};\gamma) = \mathrm{e}^{i(\alpha_1(\mathbf{r};\gamma_1)+\sigma(\mathcal{C},\Sigma))}\Psi(\mathbf{r}';\gamma), \tag{20}$$

where $U_{g_{\gamma_1}} = \exp(i\alpha_1(\mathbf{r};\gamma_1) \in \mathcal{Q}$, and $\mathbf{r}' = \mathbf{r}\gamma(1)$.

For the product of $\Psi(\mathbf{r},\gamma_1)$ and $\Psi(\mathbf{r},\gamma_2)$ we obtain

$$\Psi(\mathbf{r},\gamma_1) * \Psi(\mathbf{r},\gamma_2) = e^{i\alpha_2(\mathbf{r};\gamma_1,\gamma_2)}\Psi(\mathbf{r},\gamma_{12}), \tag{21}$$

where

$$\alpha_2(\mathbf{r};\gamma_1,\gamma_2) = e\int_\Sigma \mathbf{B}\cdot d\mathbf{S} = e\Phi\big|_\Sigma \tag{22}$$

$\Phi\big|_\Sigma$ being a magnetic flux through the two-dimensional simplex Σ with the vertices are $(\mathbf{r},\mathbf{r}_1,\mathbf{r}_2)$, where $\mathbf{r}_1 = \mathbf{r}\gamma_1(1)$ and $\mathbf{r}_2 = \mathbf{r}\gamma_2(1)$. Finally, the triple-product obeys

$$\begin{aligned} &\Psi(\mathbf{r},\gamma_1) * \big(\Psi(\mathbf{r},\gamma_2) * \Psi(\mathbf{r},\gamma_3)\big) \\ &\quad = e^{i\alpha_3(\mathbf{r};\gamma_1,\gamma_2,\gamma_3)}\big(\Psi(\mathbf{r},\gamma_1) * \Psi(\mathbf{r},\gamma_2)\big) * \Psi(\mathbf{r},\gamma_3). \end{aligned} \tag{23}$$

Returning to the AB effect, let us consider the following paths (Fig.1):

$$\gamma_1(t) : \mathbf{r} \to \mathbf{r} + t(\mathbf{a}_1 + \mathbf{a}_2), \tag{24}$$

$$\gamma_2(t) : \mathbf{r} \to \mathbf{r} + t(\mathbf{a}_3 + \mathbf{a}_4). \tag{25}$$

Now, taking into account the formula (21) one can easily see that the result of Eq.(10) is replaced by

$$\begin{aligned}\Psi_Q &= \Psi_1(\mathbf{r},\gamma_1) + \Psi_2(\mathbf{r},\gamma_2)\\ &= e^{i\alpha_1(\mathbf{r},\gamma_1)}\psi_1(\mathbf{r}) + e^{i\alpha_1(\mathbf{r},\gamma_2)}\psi_2(\mathbf{r})\\ &= e^{i\alpha_1(\mathbf{r},\gamma_2)}\big(e^{i\alpha_2(\mathbf{r};\gamma_1,\gamma_2)}\psi_1(\mathbf{r}) + \psi_2(\mathbf{r})\big), \end{aligned} \tag{26}$$

and a relative phase shift $\Delta\varphi$ is given by

$$\Delta\varphi = \alpha_2(\mathbf{r};\gamma_1,\gamma_2) = e\int_\Sigma \mathbf{B}\cdot d\mathbf{S} = e\Phi\big|_\Sigma \tag{27}$$

where $\Phi\big|_\Sigma$ is *the magnetic flux of the monopole* through the surface Σ bounded by γ_1 and γ_2. Evidently, there is *no any contribution from the Dirac string to the AB effect*.

Note, that for the true AB effect $\Phi\big|_\Sigma$ is equal to the total magnetic fluxΦ through the solenoid, and we obtain for the relative phase shift $\Delta\varphi$ the same result as in (11),

$$\Delta\varphi = \int_\Sigma \mathbf{B}\cdot d\mathbf{S} = e\oint_\gamma \mathbf{A}\cdot d\mathbf{r} = e\Phi. \tag{28}$$

We close with some comments about relevance of this work to nonassociative quantum mechanics. A standard quantum mechanics deals with a linear Hilbert space and associative operators, therefore, the Dirac quantization rule is a necessary condition for the self-consistence of quantum mechanics in the presence of magnetic monopoles. Avoiding this condition implies introducing a *nonassociative algebra of observables*. Since the nonassociativity produces a serious conflict with a Hilbert space, one must define an nonassociative equivalent to quantum mechanics, may be without Hilbert space and in terms of density matrices alone [13, 14, 15, 16, 33].

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* Electronic address: nesterov@cencar.udg.mx

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